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On the behaviour of Hodge spectral exponents of plane branches

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Resum *(CAT)*
Els exponents espectrals de Hodge són un conjunt discret d'invariants d'una singularitat aïllada d'una hipersuperfície. En aquest article estudiem la seva distribució pel cas de branques planes, en termes d'invariants numèrics de la branca. En primer lloc calculem la distribució límit per a diferents maneres de fer el límit. En segon lloc donem una fórmula tancada per a la diferència acumulada entre la distribució d'exponents espectrals de Hodge i una distribució contínua, la qual és el límit més comú. Utilitzem aquesta expressió per a obtenir intervals de valors dominants.

Abstract *(ENG)*
The Hodge spectral exponents are a discrete set of invariants of an isolated hypersurface singularity. We study their distribution for the case of plane branches, in terms of numerical invariants of the branch. First, we calculate the limit distribution for different ways of taking the limit. Secondly, we provide a closed formula for the cumulative difference of the distribution of Hodge spectral exponents with a continuous distribution, which is the most common limit. We use this expression to obtain intervals of dominating values.

Keywords: *Hodge spectral exponents, plane curve singularities, Puiseux pairs, limit distribution, characteristic function, jumping numbers.*
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1. Introduction

Let $f\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a germ of a holomorphic function (or equivalently a convergent power series $f \in \mathbb{C}\{x_0,\dots,x_n\}$) with an isolated singularity at the origin. Using the canonical mixed Hodge structure of the cohomology groups of the Milnor fiber of f, Steenbrink [7] defined the *Hodge spectrum* of f as the generating function

$$\mathrm{Sp}_f(T) = \sum_{i=1}^{\mu} T^{\alpha_i},$$

where $\mu = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_0,\dots,x_n\}}{\left(\frac{df}{dx_0},\dots,\frac{df}{dx_n}\right)}$ is the *Milnor number* and the positive rational numbers

$$0 < \alpha_1 \leqslant \cdots \leqslant \alpha_\mu < n+1$$

form a discrete set of invariants of the singularity f called *Hodge spectral exponents* (or *spectral numbers*). They are symmetric with respect to $(n+1)/2$, i.e., for every $j = 1,\dots,\mu$, we have $\alpha_{\mu+1-j} = (n+1) - \alpha_j$ and thus it is enough to study them in the interval $(0,(n+1)/2]$.

Another interesting feature proved by Varchenko [10] is that the Hodge spectral exponents of f are stable under deformations with constant Milnor number μ. A deformation of a hypersurface $f(x_0,\dots,x_n) \in \mathbb{C}\{x_0,\dots,x_n\}$ is a family of hypersurfaces $f_{t_1,\dots,t_k}(x_0,\dots,x_n)$ for some set of parameters $(t_1,\dots,t_k) \in S \subseteq \mathbb{C}^k$, satisfying $f(x_0,\dots,x_n) = f_{0,\dots,0}(x_0,\dots,x_n)$. Then, in Varchenko's result we are asking that the Milnor number of $f_{t_1,\dots,t_k}(x_0,\dots,x_n)$ is the same for all $(t_1,\dots,t_k) \in S$.

K. Saito [4] considered the normalized spectrum which he denoted as the *characteristic function*

$$\chi_f(T) = \frac{1}{\mu}\sum_{i=1}^{\mu} T^{\alpha_i}.$$

We may also display the Hodge spectral exponents as a discrete (probability) distribution on $\mathbb{R}$. Namely, the *distribution of the Hodge spectral exponents* is

$$D_f(s) = \frac{1}{\mu}\sum_{i=1}^{\mu} \delta(s-\alpha_i),$$

where $\delta(s)$ is the Dirac's delta distribution. Indeed, considering either $D_f(s)$ or $\chi_f(T)$ is equivalent because the characteristic function is the Fourier transform of the distribution of Hodge spectral exponents, i.e.,

$$\chi_f(T) = \mathcal{F}\{D_f(s)\}(\tau).$$

Considering the change of variables $T = e^{2\pi i\tau}$ we treat the dependence of χ_f on T and on τ interchangeably throughout this paper.

Remark 1.1. Because of the symmetry of the Hodge spectrum, we are interested in the truncations

$$\chi_f^{<1}(T) = \frac{1}{\mu}\sum_{\alpha_i<1} T^{\alpha_i}, \quad D_f^{<1}(s) = \frac{1}{\mu}\sum_{\alpha_i<1} \delta(s-\alpha_i).$$

Definition 1.2. The *continuous distribution* function is $N_{n+1}\colon \mathbb{R} \to \mathbb{R}$ defined as:

$$N_{n+1}(s) = \int_{x_0+\cdots+x_n=s} \mathbb{1}_{[0,1)}(x_0)\cdots\mathbb{1}_{[0,1)}(x_n)\,\mathrm{d}x_0\ldots\mathrm{d}x_n = \left(\mathbb{1}_{[0,1)} * \overset{n+1}{\cdots} * \mathbb{1}_{[0,1)}\right)(s),$$

where $\mathbb{1}_{[0,1)}(s)$ is the indicator function and $*$ denotes the convolution product. One may check that the Fourier transform of $N_{n+1}(s)$ is

$$\mathcal{F}\{N_{n+1}(s)\}(\tau) = \left(\frac{T-1}{\log T}\right)^{n+1}.$$

Definition 1.3. We define $\phi_f\colon \left[0, \frac{n+1}{2}\right) \to \mathbb{R}$ as the *cumulative difference function* between $N_{n+1}(s)$ and $D_f(s)$, that is,

$$\phi_f(r) = \int_0^r N_{n+1}(s) - D_f(s)\,\mathrm{d}s = \int_0^r N_{n+1}(s) - \frac{1}{\mu}\sum_{i=1}^{\mu}\delta(s-\alpha_i)\,\mathrm{d}s = \int_0^r N_{n+1}(s)\,\mathrm{d}s - \frac{1}{\mu}\#\{\alpha_i \leqslant r\}.$$

Definition 1.4. We say that $r \in \left[0, \frac{n+1}{2}\right)$ is a *dominating value* if $\phi_f(r) > 0$, or equivalently if

$$\frac{1}{\mu}\#\{\alpha_i \leqslant r\} < \int_0^r N_{n+1}(s)\,\mathrm{d}s.$$

K. Saito [4] introduced these notions of cumulative difference function and dominating values. Moreover he formulated the following questions:

Question 1. *For which limits of sequences of hypersurfaces $(f^{(i)})_{i\geqslant 0}$ does the distribution of Hodge spectral exponents $D_{f^{(i)}}(s)$ converge to $N_{n+1}(s)$? Equivalently, for which limits of $(f^{(i)})_{i\geqslant 0}$ does the characteristic function $\chi_{f^{(i)}}(T)$ converge to $\mathcal{F}\{N_{n+1}\}(\tau) = \left(\frac{T-1}{\log T}\right)^{n+1}$?*

Question 2. *Given f, what is the set of all dominating values?*

The limit of $(f^{(i)})_{i\geqslant 0}$ in Question 1 has to be specified, since it is not clear a priori which kind of limit one should consider. The few results we may find in the literature all consider different types of limits. K. Saito already calculated the following two limits, both of which converge to $N_{n+1}(s)$:

Proposition 1.5 ([4, (3.7)])**.** *Let $f \in \mathbb{C}[x_0,\ldots,x_n]$ be a quasi-homogeneous polynomial of degree 1 with respect to the weights $r_0,\ldots,r_n$, i.e., satisfying $f(\lambda^{r_0}x_0,\ldots,\lambda^{r_n}x_n) = \lambda f(x_0,\ldots,x_n)$. Then, taking a sequence of such functions with the limit $r_i \to 0$ for all $i = 0,\ldots,n$, one has*

$$\lim_{r_0,\ldots,r_n\to 0}\chi_f(T) = \left(\frac{T-1}{\log T}\right)^{n+1} = \mathcal{F}\{N_{n+1}(s)\}(\tau).$$

Proposition 1.6 ([4, (3.9)])**.** *Let $f\in\mathbb{C}\{x,y\}$ be an irreducible plane curve with Puiseux pairs $(n_1,l_1),\ldots,(n_g,l_g)$. Then, taking a sequence of such functions with the limit $n_g \to +\infty$ (keeping all other n_j and l_j fixed), one has*

$$\lim_{n_g\to+\infty}\chi_f(T) = \left(\frac{T-1}{\log T}\right)^{2} = \mathcal{F}\{N_2(s)\}(\tau).$$

The Puiseux pairs are defined in Section 2.

More recently, Almirón and Schulze gave another example for which the distribution of Hodge spectral exponents also converges to the continuous distribution $N_{n+1}(s)$:

Proposition 1.7 ([1]). *Consider a fixed Newton diagram Γ. Let $f_\omega \in \mathbb{C}\{x_0, \dots, x_n\}$ be a Newton non-degenerate function with Newton diagram $\omega\Gamma$ (the rescaling of Γ by a factor $\omega \in \mathbb{Q}_{>0}$). Then, taking a sequence of such functions with limit $\omega \to +\infty$, one has*

$$\lim_{\omega \to +\infty} \chi_{f_\omega}(T) = \left(\frac{T-1}{\log T} \right)^{n+1} = \mathcal{F}\{N_{n+1}(s)\}(\tau).$$

Regarding Question 2 on the set of dominating values, Tomari proved the following result which, in terms of dominating values, states the following:

Theorem 1.8 ([9]). *Let $f \in \mathbb{C}\{x, y\}$ be a plane curve. Then $\frac{1}{2}$ is a dominating value, i.e.,*

$$\#\left\{ \alpha_i \leqslant \frac{1}{2} \right\} < \frac{\mu}{8}.$$

K. Saito asked whether $\frac{1}{2}$ is a dominating value for any $f \in \mathbb{C}\{x_0, \dots, x_n\}$, that is, whether

$$\#\left\{ \alpha_i \leqslant \frac{1}{2} \right\} < \frac{\mu}{(n+1)!\, 2^{n+1}}.$$

A conjecture posed by Durfee states:

Conjecture 1.9 ([3]). *Let $f \in \mathbb{C}\{x, y, z\}$ be a surface with a singularity at the origin. Then*

$$p_g < \frac{\mu}{6}.$$

Here, p_g denotes the *geometric genus* of f defined as

$$p_g = \dim_{\mathbb{C}}(R^{n-1}\pi_* \mathcal{O}_X)_0 \text{ for } n \geqslant 2 \quad (p_g = \dim_{\mathbb{C}}(\pi_* \mathcal{O}_X / \mathcal{O}_{\mathbb{C}^2})_0 \text{ for } n = 1),$$

with $\pi\colon X \to \mathbb{C}^{n+1}$ being a resolution of the singularity. M. Saito [5] proved a relation between this invariant and the Hodge spectral exponents, namely $p_g = \#\{i \mid \alpha_i \leqslant 1\}$, and thus Durfee's conjecture predicts that 1 is a dominating value for $n = 2$. K. Saito asked whether one can generalize this statement:

Question 3. *Is 1 a dominating value for all $n \geqslant 2$? That is, is it true that*

$$p_g = \#\{\alpha_i \leqslant 1\} < \frac{\mu}{(n+1)!}$$

for any $f \in \mathbb{C}\{x_0, \dots, x_n\}$?

The aim of this work is to study Questions 1 and 2 for the case of plane branches. Regarding Question 1 we calculate the limit distribution for the limits $n_k \to +\infty$ and $l_k \to +\infty$. Regarding Question 2, we give a closed formula for $\#\{\alpha_i \leqslant r\}$ in Theorem 4.1 and $\phi_f(r)$ in Theorem 4.2 in terms of numerical invariants of the plane branch. Thereby, we provide in Theorem 5.1 intervals of dominating values.

2. Plane branch singularities

In this section we briefly present the necessary background on irreducible plane curves that we use in this paper and we refer to Casas-Alvero's book [2] for unexplained terminology.

Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C},0)$ be a germ of a holomorphic function, or equivalently a convergent power series $f \in \mathbb{C}\{x,y\}$. The equation $f=0$ defines locally a (complex) plane curve around the origin. We only consider irreducible plane curves f (also called plane branches), i.e., irreducible elements of the unique factorization domain $\mathbb{C}\{x,y\}$.

Theorem 2.1 (Puiseux)**.** *Let $f \in \mathbb{C}\{x,y\}$ define an irreducible plane curve that is not tangent to the y-axis (i.e., $\frac{\partial f}{\partial x} \neq 0$). Then there is a Puiseux series $s(x) = \sum_{i\geqslant 0} a_i x^{i/m}$ such that $f(x,s(x)) = 0$. Moreover, all such series are conjugates $\sigma_\varepsilon(s) = \sum_{i\geqslant 0} a_i \varepsilon^i x^{i/m}$ with $\varepsilon^m = 1$. The curve can be parameterized by $t \mapsto \left(t^m, \sum_{i\geqslant 0} a_i t^i\right)$.*

A Puiseux series of f has the form

$$s(x) = \sum_{\substack{j\in(e_0)\\ 0\leqslant j<\beta_1}} a_j x^{j/m} + \sum_{\substack{j\in(e_1)\\ \beta_1\leqslant j<\beta_2}} a_j x^{j/m} + \cdots + \sum_{\substack{j\in(e_{g-1})\\ \beta_{g-1}\leqslant j<\beta_g}} a_j x^{j/m} + \sum_{\substack{j\in(e_g)\\ \beta_g\leqslant j}} a_j x^{j/m}$$

with

$$e_0 = m, \quad \beta_i = \min\{j \mid a_j \neq 0,\ j \notin (e_{i-1})\}, \quad e_i = \gcd(e_{i-1},\beta_i) \quad (i=1,\dots,g),$$

where m is chosen such that $e_g = 1$. Since $e_i | e_{i-1}$, we can define $n_i = e_{i-1}/e_i \geqslant 2$.

These numerical invariants have a geometric meaning: e_0 is the multiplicity of f at the origin and e_i ($i=1,\dots,g$) is the multiplicity of f at the i-th rupture divisor of its minimal embedded resolution, or equivalently the last infinitely near point of the i-th cluster of consecutive satellite points. These concepts are explained in [2].

From the Puiseux series we can define:

Definition 2.2. The *characteristic exponents* of a plane branch f are the rational numbers $\left(\frac{\beta_1}{m},\dots,\frac{\beta_g}{m}\right)$.

Following the notation of M. Saito [6] with a slight modification, let

$$\frac{\beta_i}{m} = 1 + \frac{l_1}{n_1} + \cdots + \frac{l_i}{n_1\cdots n_i} \quad (i=1,\dots,g)$$

with $n_j \geqslant 2$, $l_j \geqslant 1$, $\gcd(l_j,n_j) = 1$. From this we define:

Definition 2.3. The *Puiseux pairs* of an irreducible plane curve f are $(n_1,l_1),\dots,(n_g,l_g)$.

The characteristic exponents and the Puiseux pairs are two equivalent sets of complete topological invariants of the singularity of f. That is: they determine, and are determined by, the homeomorphism class of $f^{-1}(0)\cap U$ for a small enough neighbourhood U of the origin.

Remark 2.4. The name Puiseux pairs appear in various slightly different ways in the literature. We based our definition on the one given by M. Saito [6], who used this name for the pairs $(k_1,n_1),\dots,(k_g,n_g)$ with $k_1 = n_1 + l_1$, $k_i = l_i$. Casas-Alvero [2] used the similar term characteristic pairs to refer to $(\beta_1,m),\dots,(\beta_g,m)$.

M. Saito [6] considered the following numerical invariants in order to obtain a formula for the characteristic function of the Hodge spectral exponents of an irreducible plane curve. We simplify the definition by letting $n_0 = 1$.

Definition 2.5. We define the following numerical invariants:

$$\begin{aligned} &w_0 = 1, \quad w_j = n_j n_{j-1} w_{j-1} + l_j && (j = 1, \dots, g),\\ &\mu_0 = 0, \quad \mu_j = (n_j - 1)(w_j - 1) + n_j \mu_{j-1} && (j = 1, \dots, g). \end{aligned}$$

Proposition 2.6 ([4]). *The Milnor number of f is $\mu = \mu_g$. More generally, the Milnor number of a curve with Puiseux pairs $(n_1, l_1), \dots, (n_j, l_j)$ is μ_j, for any $j \in \{1, \dots, g\}$.*

From these definitions we prove:

Lemma 2.7. *The Milnor number of an irreducible plane curve with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$ is*

$$\mu = \sum_{j=1}^{g} l_j e_j (e_{j-1} - 1) + (e_0 - 1)^2.$$

Thành and Steenbrink [8] already described the Hodge spectrum of any plane curve in terms of its topological invariants, but in this work we use a closed formula given by M. Saito:

Theorem 2.8 ([6, Theorem 1.5]). *The Hodge spectral exponents in the interval $(0, 1)$ are:*

$$\left\{ \frac{1}{e_j}\left(\frac{b}{n_j} + \frac{a}{w_j}\right) + \frac{c}{e_j} \;\middle|\; \begin{array}{l} 0 < a < w_j \\ 0 < b < n_j \\ 0 \leqslant c < n_{j+1} \cdots n_g \\ 1 \leqslant j \leqslant g \end{array}, \; \frac{b}{n_j} + \frac{a}{w_j} < 1 \right\}.$$

Notice that this formula gives us a set of $\mu/2$ Hodge spectral exponents and thus, by symmetry, it characterizes all the Hodge spectral exponents of f.

To work with characteristic functions (i.e., Fourier transforms), M. Saito defined:

Definition 2.9. Let $F(T) = \sum_{i \geqslant 0} a_i T^{i/N} \in \mathbb{C}[T^{1/N}]$. Then, we define the following truncations:

$$F^{<1}(T) = \sum_{i/N<1} a_i T^{i/N}, \quad F^{>1}(T) = \sum_{i/N>1} a_i T^{i/N},$$

which are the terms of $F(T)$ with exponents smaller and larger than 1 respectively.

Definition 2.10. We define the auxiliary functions $\Phi_j(T)$ as:

$$\Phi_1(T) = \frac{T^{1/w_1} - T}{1 - T^{1/w_1}} \frac{T^{1/n_1} - T}{1 - T^{1/n_1}},$$

$$\Phi_j(T) = \frac{1 - T}{1 - T^{1/n_j}} \Phi_{j-1}^{<1}(T^{1/n_j}) + T^{1-1/n_j} \frac{1 - T}{1 - T^{1/n_j}} \Phi_{j-1}^{>1}(T^{1/n_j}) + \frac{T^{1/w_j} - T}{1 - T^{1/w_j}} \frac{T^{1/n_j} - T}{1 - T^{1/n_j}}$$

(for $j = 2, \dots, g$).

Then, M. Saito proves the following theorem:

Theorem 2.11 ([6, Theorem 1.5]). *The characteristic function of an irreducible plane curve f is*

$$\chi_f(T) = \frac{1}{\mu_g} \Phi_g(T).$$

3. Limit distribution in the case of branches

In this section we study the case of plane branches for K. Saito's Question 1 on the limit distribution of Hodge spectral exponents. We consider irreducible plane curves $f \in \mathbb{C}\{x, y\}$ with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$. In this case, K. Saito's Question 1 asks for which limits of irreducible plane curves f does the distribution of Hodge spectral exponents $D_f(s)$ converge to $N_2(s)$ (recall Definition 1.2). Equivalently, it asks for which limits of the Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$ does the characteristic function $\chi_f(T) = \mathcal{F}\{D_f(s)\}(\tau)$ converge to the Fourier transform $\mathcal{F}\{N_2(s)\}(\tau) = \left(\frac{T-1}{\log T}\right)^2$.

K. Saito computed the particular case of a limit where the invariant n_g of the last Puiseux pair tends to infinity while all the remaining Puiseux pairs are kept fixed. His result, recalling Proposition 1.6, is that the resulting limit distribution of Hodge spectral exponents is

$$\lim_{n_g \to +\infty} \chi_f(T) = \left(\frac{T-1}{\log T}\right)^2 = \mathcal{F}\{N_2\}(\tau).$$

This is the expected limit distribution of Question 1. Given this result, we are led to ask whether it is possible to generalize it for the following limits:

(i) $n_k \to +\infty$ for a particular $k \in \{1, \dots, g\}$ while keeping all other n_j and l_j fixed,

(ii) $l_k \to +\infty$ for a particular $k \in \{1, \dots, g\}$ while keeping all other n_j and l_j fixed.

For the first case we prove the following:

Theorem 3.1. *Let $g \in \mathbb{Z}_{>0}$, $k \in \{1, \dots, g\}$. Let f be an irreducible plane curve with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$. Consider a sequence of such curves f with $n_k \to +\infty$, n_j $(j \neq k)$ fixed and all l_j fixed. Then, the limit of the characteristic function is*

$$\lim_{n_k \to +\infty} \chi_f(T) = \left(\frac{T-1}{\log T}\right)^2.$$

Equivalently, the limit of the distribution of Hodge spectral exponents is

$$\lim_{n_k \to +\infty} D_f(s) = N_2(s).$$

The preceding theorem states that sequences of irreducible plane curves with $n_k \to +\infty$ (with the other numerical invariants fixed) form a family of solutions to K. Saito's Question 1 on the limit distribution of Hodge spectral exponents.

On the other hand, we prove:

Theorem 3.2. *Let $g \in \mathbb{Z}_{>0}$, $k \in \{1, \dots, g\}$. Let f be an irreducible plane curve with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$. Consider a sequence of such curves f with $l_k \to +\infty$, l_j $(j \neq k)$ fixed and all n_j fixed. Then, the limit of the characteristic function is*

$$\lim_{l_k \to +\infty} \chi_f(T) = \frac{1}{e_{k-1} - 1} \frac{T-1}{\log T} \frac{T^{1/e_{k-1}} - T}{1 - T^{1/e_{k-1}}}.$$

Equivalently, the limit of the distribution of Hodge spectral exponents is

$$\lim_{l_k \to +\infty} D_f(s) = \frac{1}{e_{k-1} - 1} \left(\lfloor e_{k-1} s \rfloor \mathbb{1}_{[0,1)}(s) + \lfloor e_{k-1}(2 - s) \rfloor \mathbb{1}_{[1,2)}(s) \right),$$

where $\mathbb{1}_{[a,b)}(s)$ denotes the indicator function of the interval $[a, b)$.

These limits are different from $\left(\frac{T-1}{\log T}\right)^2$ and $N_2(s)$ respectively. Therefore, this theorem says that sequences of irreducible plane curves with $l_k \to +\infty$ (with the other numerical invariants fixed) are a family of *non*-solutions to K. Saito's Question 1 on the limit distribution of Hodge spectral exponents.

4. Cumulative difference function ϕ_f

From Definition 1.3 we have that the cumulative difference function for the case of plane curves is $\phi_f\colon [0,1) \to \mathbb{R}$ defined as

$$\phi_f(r) = \int_0^r N_2(s) - D_f(s)\,\mathrm{d}s = \int_0^r N_2(s) - \frac{1}{\mu}\sum_{i=1}^{\mu}\delta(s-\alpha_i)\,\mathrm{d}s = \frac{1}{2}r^2 - \frac{1}{\mu}\#\{\alpha_i \leqslant r\}$$

since we have $N_2(s) = s$ in the interval $[0,1)$.The Hodge spectral exponents α_i are given by Theorem 2.8.

To be able to study the set of dominating values (i.e., K. Saito's Question 2) we need a more explicit expression for $\#\{\alpha_i \leqslant r\}$ or equivalently $\phi_f(r)$. For this purpose we prove the following:

Theorem 4.1. *Let f be an irreducible plane curve with Puiseux pairs $(n_1,l_1),\dots,(n_g,l_g)$. Then, for any $r \in [0,1)$, the number of Hodge spectral exponents less or equal to r is given by the following expression:*

$$\begin{aligned}\#\{\alpha_i \leqslant r\} = {} & \frac{\mu_g - n_g w_g}{2}r + \frac{n_g w_g}{2}r^2 + \sum_{j=1}^{g}\frac{n_j-1}{2}\{e_j r\} + \frac{1}{2}\{e_0 r\}(1-\{e_0 r\}) \\ & + \sum_{j=1}^{g}\frac{l_j}{2n_j}\{e_{j-1}r\}(1-\{e_{j-1}r\}) - \sum_{j=1}^{g}\sum_{b=1}^{n_j-1}\left\{w_j\left(\{e_j r\} - \frac{b}{n_j}\right)\right\}\mathbb{1}_{\left[\frac{b}{n_j},1\right)}(\{e_j r\}).\end{aligned}$$

Theorem 4.2. *Let f be an irreducible plane curve with Puiseux pairs $(n_1,l_1),\dots,(n_g,l_g)$. Then, for any $r \in [0,1)$, the cumulative difference function between $N_2(s)$ and $D_f(s)$ is given by the following expression:*

$$\begin{aligned}\phi_f(r) = \frac{1}{2\mu}\Bigg(& \left(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j\right) r(1-r) - \sum_{j=1}^{g}(n_j-1)\{e_j r\} - \{e_0 r\}(1-\{e_0 r\}) \\ & - \sum_{j=1}^{g}\frac{l_j}{n_j}\{e_{j-1}r\}(1-\{e_{j-1}r\}) + \sum_{j=1}^{g}\sum_{b=1}^{n_j-1} 2\left\{w_j\left(\{e_j r\} - \frac{b}{n_j}\right)\right\}\mathbb{1}_{\left[\frac{b}{n_j},1\right)}(\{e_j r\})\Bigg).\end{aligned}$$

5. Dominating values for irreducible plane curves

In this section we give partial answers to K. Saito's Question 2 on the dominating values for the case of irreducible plane curves. To such purpose we use Theorem 4.2, which gives us an explicit expression for the cumulative difference function $\phi_f(s)$. This expression can be used to find simpler functions which bound $\phi_f(s)$, thus making it possible to obtain intervals where ϕ_f is positive. We prove the following result:

Theorem 5.1. *Let f be an irreducible plane curve with Puiseux pairs $(n_1, l_1), \dots, (n_g, l_g)$. Then,*

(i) A set of dominating values is given by the interval

$$r \in \left(\frac{\left(2e_0 - n_g + \sum_{j=1}^{g} l_j e_j\right) - \sqrt{D_1}}{2\left(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j\right)}, \frac{\left(2e_0 - n_g + \sum_{j=1}^{g} l_j e_j\right) + \sqrt{D_1}}{2\left(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j\right)} \right)$$

with

$$D_1 = \left(2e_0 - n_g + \sum_{j=1}^{g} l_j e_j\right)^2 - 4\left(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j\right)\left(\sum_{j=1}^{g-1}(n_j - 1) + \frac{1}{4} + \sum_{j=1}^{g} \frac{l_j}{4n_j}\right) > 0.$$

(ii) A set of dominating values is given by the interval

$$r \in \left(\frac{\left(e_0 + \sum_{j=1}^{g} l_j e_j\right) - \sqrt{D_2}}{2\left(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j\right)}, \frac{\left(e_0 + \sum_{j=1}^{g} l_j e_j\right) + \sqrt{D_2}}{2\left(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j\right)} \right)$$

with

$$D_2 = \left(e_0 + \sum_{j=1}^{g} l_j e_j\right)^2 - 4\left(2e_0 - 1 + \sum_{j=1}^{g} l_j e_j\right)\left(\frac{1}{4} + \sum_{j=1}^{g} \frac{l_j}{4n_j}\right) > 0.$$

(iii) We have that the leftmost interval of $(0, 1)$

$$r \in (0, \mathrm{lct}(f)) = \left(0, \frac{1}{e_1}\left(\frac{1}{n_1} + \frac{1}{n_1 + l_1}\right)\right)$$

is a set of dominating values. In contrast, $\phi_f(r) < 0$ for the rightmost interval

$$r \in \left[1 - \frac{1}{n_g w_g}, 1\right).$$

We point out that the first two intervals always intersect but it is not always clear which are the ends of the unique interval of dominating values they provide. The intervals of the third item are obtained directly from the smallest and largest Hodge spectral exponents.

Remark 5.2. Almirón and Schulze [1, Proposition 6] proved that the log-canonical threshold of an irreducible plane curve is a dominating value except for the cases where the curve has semigroup $(2, 3)$ or $(2, 5)$.

In the course of the proof of Theorem 5.1 we also obtain an alternative proof of Theorem 1.8 by Tomari [9] but only for irreducible curves.

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